Technical Paper:

Valuation of Credit Derivatives

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1 A Copula based Monte Carlo method for the valuation of basket credit linked derivatives

Within the UnRisk PRICING ENGINE, the pricing of basket credit derivatives is performed using a copula based Monte Carlo framework. The instruments which belong to the class of multiname credit derivatives are Nth to default notes, Nth to default swaps, basket credit linked notes and collaterized debt obligations (CDOs). While Gaussian Copulas are widely used in praxis, the default risk is strongly underestimated by the tails of the normal distribution. The choice of the underlying copula is an important issue in the valuation of basket credit linked derivatives. Alternative to the simple gaussian copula, a Student t copula (using different degrees of freedom) can be used to model the default dependencies in the UnRisk PRICING ENGINE.

1.1 Introduction

A variety of modern credit derivatives is based on an underlying basket of bonds of different obligors. Using a valid model for the default correlations is crucial to the pricing of these instruments. A very challenging problem in the valuation procedure of basket credit derivatives is the modelling of joint default times of the underlying assets. Since default probabilities are mostly small, the impact of the correlation structure is much higher than for euity, fx or interest rate derivatives. The basic idea of the copula simulation approach is to split the time structure of defaults from the dependency structure between the underlying obligors. This means that the default structure can be calibrated for each underlying name separately. This seperation approach can span the whole variety of possible dependencies. The mathematical key concept hereto are copulae.

1.2 Copulas

Definition:

A function $C : [0,1]^I \to [0,1]$, $I \in \mathbb{N}$ is a copulae, if

1. there are random variables $U_1, ..., U_I$ taking values in $[0,1]$ such that C is their joint distribution fucntion.

2. C has uniform marginal distributions, i.e. $C(1, .., 1, u_i, 1, .., 1) = u_i$ for all $i \leq I, u_i \in [0,1]$

There is an important theorem by Sklar stating that for any multivariate distribution function the marginal distributions (which represents the time dependency of a single name default) and the dependence structure (not time dependent) can be separated. In other words, from the joint distribution we can construct a copulae which allows us to reassemble the joint distribution from the marginal distributions.

Sklars Theorem:

Let $X_1, ..., X_n$ be random variables with marginal distribution functions $F_1, ..., F_n$ and joint distribution function F. Then there exists an n-dimensional copula C such that for all x $\in \mathbb{R}^n$:

$$F(x) = C(F_1(x_1), ..., F_n(x_n)) = C(\mathbf{F}(\mathbf{x}))$$

If $F_1, ..., F_n$ are continuous, then C is unique.

Note that by construction a copulae always leads to the desired marginal distributions. For credit derivatives pricing this means, that whatever copula we chose to model dependency between defaults, this leaves the defualt distribution for the single name unchanged.

1.2.1 Gaussian Copulae

Let Φ_R denote the multivariate student t distribution function with ν degrees of freedom and a correlation matrix R, let its mean be zero in all components and the variances be equal to one. Thus, the marginal distributions are all standard univariate normal distributions. Then from Sklars theorem we know that there exists a copulae C_R with $C_R(\Phi(x_1), ..., \Phi(x_n)) = \Phi_R(x_1, ..., x_n)$. A simple transformation of the variables $u_i = \Phi(x_i)$ yields $C_R(u_1, ..., u_n) = \Phi_R(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n))$. This copulae C_R is called the Gaussian copulae with correlation matrix R.

1.2.2 Generation of Random Variates from a Gaussian Copulae

1. Calculate the Cholesky decomposition of the correlation matrix R, i.e. $R = AA^T$
2. Simulate n independent standard normal random variables w=$(w_1, ..., w_n)$
3. Calculate a random vector using $x = Aw$.
4. Transform x to a vector u of uniform distributed random variables on $[0, 1]$ using $u = \Phi(x)$, with Φ denoting the cumulative univariate standard normal distribution funciton. The resulting vector u is a random variate from the n- dimensional Gaussian copulae C_R

1.2.3 Student t Copulae

Let $t_{\nu,R}$ denote the multivariate normal distribution function with correlation matrix R, let its mean be zero in all components and the variances be equal to one. Chosing a random vector X following a multivariate normal distribution with correlation matrix R, means zero and variances one and chosing W to follow a χ^2 distribution with ν degrees of freedom. Then the distribution of $\frac{\sqrt{\nu}}{\sqrt{W}}X$ is $t_{\nu,R}$, i.e. the standardized mulitvariate student t distribution with marginal distributions t_ν. The copula $C_{\nu,R}(u_1, ..., u_n) = t_{\nu,R}(t_n u^{-1}(u_1), ..., t_n u^{-1}(u_n))$ is called the student t copulae. The student t copulae cannot represent the case of independent obligors. By definition there is some tail dependence which always makes it more likely that two obligors both default or both survive.

1.2.4 Generation of Random Variates from a Studen t Copulae

1. Calculate the Cholesky decomposition of the correlation matrix R, i.e. $R = AA^T$

2. Simulate n independent standard normal random variables w=$(w_1, ..., w_n)$
3. Simulate a random variate s which is independent of w, following a χ^2 distribution with ν degrees of freedom
4. Calculate a random vector using $y = Aw$.
5. Calculate a random vector using $x = y\sqrt{\frac{\nu}{s}}$
6. Transform x to a vector u of uniform distributed random variables on $[0, 1]$ using $u = t_\nu(x)$, with t_ν denoting the cumulative student t distribution funciton. The resulting vector u is a random variate from the n-dimensional Student t copulae $C_{\nu,R}$

1.3 Modelling joint default times

The challenge in pricing basket credit linked instruments and CDO tranches is the modelling of the joint default times. Copulae functions are due to its simplicity and fast computation widely used to model the dependence structure of baskets. The default time τ_i for every underlying in the basket is a random variable which can be modelled as a stopping time. Denoting $F_i(t) = P(\tau_i \leq t)$ the distribution function and $f_i(t)$ the density function of the stopping time, the hazard rate process $h_i(t)$ of τ_i is defined as the probability of default of the i th underlying until time $t + \Delta t$ under the condition of survival until time t, is given by

$$h_i(t) = \underset{\Delta t \to 0}{lim} \frac{P(t < \tau_i < t + \Delta t | \tau_i > t)}{\Delta t} = \frac{f_i(t)}{1 - F_i(t)}.$$

With $S_i(t) = 1 - F_i(t)$ denoting the survival function that gives the probability that the default time of the i th underlying is after time t, the hazard rate fucntion gives the instantaneous default probability under the condition of survival of time t. The survival function can be written as

$$S_i(t) = 1 - F_i(t) = e^{-\int_0^t h_i(s)\, ds}$$

and the distribution function as

$$F_i(t) = 1 - e^{-\int_0^t h_i(s)\, ds}.$$

The default times are defined as

$$\tau_i := inf\{t >= 0 : \int_0^t h_i(s)\, ds >= \theta_i\}$$

where θ_i has an exponential distribution with unit intensity. The joint distribution of default times is defined as

$$F(t_1, ..., t_n) = P(\tau_1 \leq t_1, ..., \tau_n \leq t_n)$$

and the joint survival time distribution as

$$S(t_1, ..., t_n) = P(\tau_1 > t_1, ..., \tau_n > t_n) = C(e^{-\int_0^t h_1(s)\, ds}, ..., e^{-\int_0^t h_n(s)\, ds})$$

Using $\lambda_i(t) := e^{-\int_0^t h_i(s)\, ds}$, the times of default τ_i are defined as the first time of any λ_i reaching the sampled correlated uniform trigger variables U_i, i.e.:

$$\tau_i := inf\{t \geq 0 : \lambda_i(t) \leq U_i\}$$

with a copula function C

$$C(u_1, ..., u_n) = P(U_1 \leq u1, ..., U_n \leq un)$$

The copulae based approach one first generates correlated random numbers X_i. Then uniformly distributed random variates U_i are calculated using the cumulative distribution function. Finally, the calculation of the default times for each underying i is done by an inverse mapping of their marginal distributions, $\tau_i = F_i^{-1}(U_i)$.

1.4 Valuation of credit linked instruments

For liquid names CDS spreads are quoted and can be used to determine the hazard rate functions $h_i(t)$ and thus the survival and default probabilities. This is done by a bootstrapping algorithm under the assumption that the hazard rates are constant between premium payment dates of the CDS. Knowing how to simulatite the joint default times, the valuation of credit linked instruments can be done performing the follwing steps. For each path k of the MC simulation simulate the default time vector $\tau_k = (\tau_1, ..., \tau_n)$ and calculate the fair value of the credit linked instrument V_k (i.e. the sum of the discounted payments). Using M Monte Carlo paths, the fair value of the credit linked instrument is finally calculated as the mean of the fair values in each Monte Carlo path, i.e.

$$V = \frac{1}{M} \sum_{i=1}^{M} V_k$$

References

[1] Fathi, Abid and Nader, Naifar:"Copula based simulation procedures for pricing basket credit derivatives", University of Sfax 2007